On the number of trees in Z^d

1992 J. Phys. A: Math. Gen. 25 3523

(http://iopscience.iop.org/0305-4470/25/12/014)

J. Phys. A: Math. Gen. 25 (1992) 3523–3528. Printed in the UK

On the number of trees in $\mathcal{Z}^d$

E J Janse van Rensburg†
Department of Chemistry, University of Toronto, Toronto, Ontario M5S 1A1, Canada

Received 29 November 1991, in final form 17 March 1992

Abstract. The number of trees weakly embedded in the hypercubic lattice, t_n, is considered. It is known that $\lim_{n\to\infty} t_n^{1/n} = \lambda_0$, and that $t_n \leqslant \lambda_0{}^n$. These facts are proven by noting that trees satisfy a supermultiplicative inequality $t_n t_m \leqslant t_{n+m}$. In this paper a *submultiplicative* property is derived for trees of the form $t_{n+m} \leqslant (n+m)^{3\alpha \log(n+m)} t_n t_m$. Consequently, there exists a constant δ such that $\mathrm{O}(\exp[-\delta(\log n)^2]\lambda_0{}^n \leqslant t_n \leqslant \lambda_0{}^n$.

1. Introduction

The number of walks embedded in the hypercubic lattice, c_n, has been the subject of numerous studies. Hammersley and Morton (1954) proved the existence of a *connective constant* κ, such that $\lim_{n\to\infty} c_n^{1/n} = \mathrm{e}^\kappa$. The connective constant is also called the *conformational entropy* (Whittington and Soteros 1991) and e^κ $(=\mu)$ is called the *growth constant*. c_n is a submultiplicative function, i.e. $c_n c_m \geqslant c_{n+m}$. Consequently, by the theory of subadditive functions, one notes that $c_n \geqslant \mu^n = \mathrm{e}^{\kappa n}$ (Hille 1948). A remarkable effort by Hammersley and Welsh (1962) established that

$$\mu^n \leqslant c_n \leqslant \mu^n\, \mathrm{O}(\mathrm{e}^{\gamma\sqrt{n}}) \tag{1.1}$$

where γ is a positive constant. Equation (1.1) rigorously limits the deviation of c_n from pure exponential growth. Kesten (1964) improved slightly on (1.1). It is widely believed that c_n deviates from exponential growth at most by a power, in analogy with magnetism (of which it is the 'zeroth component' limit). A remarkable effort by Hara and Slade (1991a) (using the lace expansion) established that $c_n = A\mu^n[1+\mathrm{O}(n^{-\epsilon})]$ in *five and more dimensions*, where $\epsilon \leqslant \frac{1}{2}$ is any fixed positive real number.

In investigations of the function u_n, the number of self-avoiding polygons in the hypercubic lattice proved generally more successful. It is easily seen that $u_n u_m \leqslant (d-1)u_{n+m}$, so that $u_n/(d-1)$ is a supermultiplicative function. (Here, d is the number of dimensions, and n in u_n takes only even values.) Consequently, one can show that $\lim_{n\to\infty}[u_n/(d-1)]^{1/n} = \mu$ exists, and has the same value as the growth constant for walks (Hammersley 1961). In addition, $u_n \leqslant (d-1)\mu^n$. A lower bound on the number of polygons was computed by Kesten (1964), and an improved upper bound calculated by Madras (1991) established that

$$\mathrm{O}(n^{-6(d-1.5)})\mu^n \leqslant u_n \leqslant \mathrm{O}(n^q)\mu^n \qquad \text{where} \quad \begin{cases} q \leqslant -\frac{1}{2} & \text{if } d=2 \\ q \leqslant -1 & \text{if } d=3 \\ q < -1 & \text{otherwise}. \end{cases} \tag{1.2}$$

† Email address: bvanrens@alchemy.chem.utoronto.ca

These results limit the deviation from pure exponential growth (of u_n) to at most a power law, but the real challenge would be to prove that $\lim_{n\to\infty} [\log(u_n/\mu^n)]/\log n$ exists.

In this paper we study the function t_n, the number of trees weakly embedded in the hypercubic lattice. The objective is to find limits on the deviation from pure exponential behaviour with n, such as was done for walks in (1.1) and polygons in (1.2). t_n is a supermultiplicative function: $t_n t_m \leqslant t_{n+m}$ (Klein 1981). Consequently, there exists a growth constant λ_0 such that

$$\lim_{n\to\infty} t_n^{1/n} = \lambda_0 . \tag{1.3}$$

Moreover,

$$t_n \leqslant \lambda_0{}^n . \tag{1.4}$$

Again, there is strong evidence that the deviation of t_n from pure exponential behaviour is bound by a power, i.e. $t_n \sim Cn^{-\theta}\lambda_0{}^n$ (where $a_n \sim f(n)$ means $a_n = f(n)[1 + \mathrm{O}(1)]$), and where C is a constant and θ is a critical exponent. (For numerical evidence see the calculations by Glaus (1985), Caracciolo and Glaus (1985), Adler *et al* (1988), Ishinabe (1989).) In *more than eight dimensions* Hara and Slade (1991b, 1992) made this rigorous, again using the lace expansion. θ is believed to be equal to 1 in two dimensions, and $\frac{3}{2}$ in three dimensions, as calculated from the dimensional reduction of Parisi and Sourlas (1981). In this paper we show that there exists a positive constant δ such that

$$\mathrm{O}(\mathrm{e}^{-\delta(\log n)^2})\lambda_0{}^n \leqslant t_n \leqslant \lambda_0{}^n . \tag{1.5}$$

Equation (1.5) is not as strong as that of Hara and Slade (1991b, 1992), but is valid in every number of dimensions more than or equal to two, in contrast to the lace-expansion results, which are valid only in more than eight dimensions. The lower bound in (1.5) is a direct result of a submultiplicative property of t_n, which is proved in section 2. In section 2 we also prove the main result, and we conclude the paper with a few remarks in section 3.

2. A submultiplicative inequality

The *bottom vertex* and the *top vertex* of a tree T_n with n edges is defined as the first and the last vertices found in a lexicographic ordering of the vertex set of T_n. A fundamental construction performed on trees is concatenation (Klein 1981). Let T_n be a tree with n edges, and top vertex t_T, and let S_m be a tree with m edges and bottom vertex b_S. T_n and S_m are concatenated by identifying t_T with b_S. The result is a tree $T_n \oplus S_m$ with $n+m$ edges. Since the construction is a one-to-one map, one finds the supermultiplicative property for t_n, the number of lattice trees (modulo a translation in the hypercubic lattice): $t_n t_m \leqslant t_{n+m}$.

A *branch* of a tree T_n is itself a tree S_m which is joined to the rest of T_n at a single vertex v. If one removes the branch from T_n, then the result is again a tree $T_n - S_m$ with $n - m$ edges. These definitions lead naturally to the following proposition.

Proposition 1. Let T_n be a tree, and let m ($\leqslant n$) be a positive integer. Then there is a branch S_q of T_n, such that $\lceil m/2d \rceil \leqslant q \leqslant m$.

Proof. If $m = n$, then T_n is the branch; so suppose that $m < n$. Let p_1 be any vertex in T_n. Incident on p_1 are $2d$ branches of T_n, and let the ith branch contain u_i edges. (Some of the branches could of course have no edges; it makes the analysis easier if one thinks of these as 'empty' branches). Then, $\sum_{i=1}^{2d} u_i = n$. Suppose that $u_i < m/2d$, $\forall i$. Then $n = \sum_{i=1}^{2d} u_i < m < n$. This is a contradiction, so there exists an i such that $u_i \geqslant m/2d$. Observe that u_i is always an integer, hence, $u_i \geqslant \lceil m/2d \rceil$.

If $u_i \leqslant m$, then the proposition is proven. So suppose that $u_i > m$. Let this branch with u_i edges be B^1 and suppose that it has b_1 ($= u_i$) edges. We now show that one can find a branch of B^1, B^2 with b_2 edges, such that $b_1 > b_2 \geqslant \lceil m/2d \rceil$. If one iterates this procedure, then finally one must find a branch B^j with b_j edges such that $m \geqslant b_j \geqslant \lceil m/2d \rceil$. Only a finite number of iterations are necessary; T_n is a finite object.

Let p_2 be any vertex in B^1, such that $p_2 \neq p_1$, and let the branch of B^1 incident on p_2 which excludes p_1, be E. (There are $2d - 1$ branches of B^1 incident on p_2, excluding the branch which contains p_1, and together they form E.) Suppose that E consists of branches incident on p_2 which have v_i edges each, for $1 \leqslant i \leqslant 2d - 1$ (i.e. index the branches with $1 \leqslant i \leqslant 2d - 1$). Then, $\sum_{i=1}^{2d-1} v_i < b_1$, since at least one edge (incident on p_1) is not counted under the summation. There are two cases to consider: case 1 has $\sum_{i=1}^{2d-1} v_i \geqslant m$. Observe then that there exists an i such that $v_i \geqslant \lceil m/(2d-1) \rceil \geqslant \lceil m/2d \rceil$; let this ith branch be B^2, and $b_2 = v_i$. Observe then that $b_1 > b_2 \geqslant \lceil m/2d \rceil$.

Case 2 has $\sum_{i=1}^{2d-1} v_i < m$. Here, there are two subcases: case 2a has $\sum_{i=1}^{2d-1} v_i \geqslant \lceil m/2d \rceil$. Observe that in this case E is a branch in the desired size range. Case 2b has $\sum_{i=1}^{2d-1} v_i < \lceil m/2d \rceil$. Here, one must add edges to E until it has the desired size. The first edge to be added is incident on p_2; it is the first edge on the shortest path between p_2 and p_1. This augmented branch E^1 is rooted in T_n at a new vertex p_2^1. Incident on p_2^1 are $2d - 2$ branches of B^1 (none of which contains p_1). Augment E^1 by adding these branches to it one by one. There are three possible outcomes: (i) after the last branch is added to p_2^1, the augmented branch is still too small. In this case rename it E^1, and repeat the construction (i.e. add an edge on the shortest path between p_2^1 and p_1 to the augmented branch, and consider the new set of $2d - 2$ branches); (ii) one might add a branch to E^1 which will increase its size from below $\lceil m/2d \rceil$ to above m. In this case the branch one adds has at least $m - \lceil m/2d \rceil \geqslant \lceil m/2d \rceil$ edges. Name this branch B^2, and observe that it has b_2 edges, where $b_1 > b_2 \geqslant \lceil m/2d \rceil$ (since p_1 is not a vertex in this branch); (iii) the augmented branch E^1 is in the desired size range after a number of branches have been added at p_2^1. Situation (i) cannot be repeated indefinitely, and eventually must lead to (ii) or (iii), since B^1 is finite. This proves the proposition. □

Naturally, proposition 1 results in the following corollary.

Corollary 1. Let T_n be a tree and let m be any positive integer such that $m \leqslant n$. Then k branches $\{B^i\}_{i=1}^k$ (where B^i is a branch with b_i edges) can be pruned from T_n such that

(i) $\sum_{i=1}^k b_i = m$,

(ii) $m - \sum_{i=1}^{l} b_i \geqslant b_{l+1} \geqslant \lceil (m - \sum_{i=1}^{l} b_i)/2d \rceil$, and
(iii) $k \leqslant (\log m)/\log[2d/(2d-1)]$.

Proof. The aim is to prune k branches from T_n such that a total of m edges are removed. The worst possible situation arises if one manages to prune the least number of edges from the tree at every step. That is, if p edges are to be removed, then only $\lceil p/2d \rceil$ are removed. Under these circumstances, suppose that l edges must be removed at the jth step in the construction. Then the branch B^j is pruned and $b_j \geqslant \lceil l/2d \rceil$ is the number of edges in B^j. At the $(j+1)$th step, at most $l - \lceil l/2d \rceil = \lfloor (2d-1)l/2d \rfloor$ edges remain to be removed. If this formula is iterated, then one notes that at the jth step, at most $\lfloor (2d-1) \lfloor (2d-1) \lfloor \cdots \lfloor (2d-1)m/2d \rfloor \cdots \rfloor /2d \rfloor /2d \rfloor$ remain to be removed (there are j factors in the expression). After k branches are removed, a total of m edges have been taken out. This occurs when the above expression attains the value 1 after k iterations, and 0 after $k+1$ iterations. But $m[(2d-1)/2d]^k \geqslant \lfloor (2d-1) \lfloor (2d-1) \lfloor \cdots \lfloor (2d-1)m/2d \rfloor \cdots \rfloor /2d \rfloor /2d \rfloor = 1$, or $k \leqslant \log m / \log[2d/(2d-1)]$. This proves (iii). (i) is easily seen: observe that at least one edge can be removed at any stage, and proposition 1 indicates that it is never necessary to remove more edges than necessary. If at the jth step not enough edges edges are taken out, then proposition 1 is applied again. (ii) is a direct consequence of proposition 1. This proves the corollary. □

Corollary 1 contains enough information to allow one to prove a submultiplicative property for t_n.

Theorem 1. t_n satisfies the following inequality in any number of dimensions $d \geqslant 2$:

$$t_{n+m} \leqslant (n+m)^{3\alpha \log(n+m)} t_n t_m$$

where $\alpha = 1/\log[2d/(2d-1)]$.

Proof. Let T_{n+m} be any tree. Apply corollary 1 to prune at most $\alpha \log m$ (where $\alpha = 1/\log[2d/(2d-1)]$) branches containing m edges from the tree. Any one of these branches can be put back at most $(n+m)m$ ways into the tree. Concatenate the branches (as they are pruned) into a new tree containing m edges. The total number of ways one can cut the tree to find the original branches is at most $m^{\alpha \log m}$. Hence, $t_{n+m} \leqslant [(n+m)m^2]^{\alpha \log m} t_n t_m$. This is better than the claimed inequality. □

We can now prove (1.5) immediately.

Theorem 2. The number of trees weakly embedded in the hypercubic lattice, t_n, in d dimensions, is bound by

$$\mathrm{e}^{-24\alpha} n^{-24\alpha} \mathrm{e}^{-9\alpha(\log n)^2} \lambda_0{}^n \leqslant t_n \leqslant \lambda_0{}^n$$

where $\alpha = 1/\log[2d/(2d-1)]$.

Proof. This result is a direct consequence of theorem 2 and a property of submultiplicative functions: if $t_{n+m} \leqslant g(n+m) t_n t_m$, then $(\log t_n)/n \geqslant \log \lambda_0 + [\log g(n)]/n - 4\sum_{m=2n}^{\infty} [\log g(m)]/[m(m+1)]$ (Hammersley 1962). Observe that

$\log g(n) = 3\alpha(\log n)^2$ (theorem 1). Here, λ_0 is the growth constant. The last step is to bound the infinite sum. We do that by an integral:

$$\sum_{m=2n}^{\infty} \frac{(\log m)^2}{m(m+1)} \leqslant \int_{2n}^{\infty} \frac{[\log(x-1)]^2}{x(x-1)} \mathrm{d}x \leqslant \int_{n}^{\infty} \left(\frac{\log x}{x}\right)^2 \mathrm{d}x .$$

The integral can be computed using standard tables (Gradshteyn and Ryznik 1965). Substituting the result into the original expression gives $(\log t_n)/n \geqslant \log \lambda_0 - 9\alpha(\log n)^2/n - 24\alpha(\log n)/n - 24\alpha/n$. Exponentiate this to find the lower bound. The upper bound is a direct consequence of supermultiplicativity ($t_n t_m \leqslant t_{n+m}$). □

To find (1.5), put $\delta = 9\alpha$.

3. Conclusions

Weakly embedded trees can be thought of as (weakly embedded) lattice animals with cyclomatic index zero. c-animals are lattice animals with cyclomatic index c uniformly weighted (on the number of edges). Let $a_n(c)$ be the number of c-animals with cyclomatic index c and n edges. Then $t_n = a_n(0)$. It is rigorously known that (Soteros and Whittington 1988)

$$A\binom{\epsilon n}{c} a_n(0)/3^c \leqslant a_{n+c}(c) \qquad (3.1)$$

if n is large enough and where $0 < \epsilon \leqslant C$; A and C are fixed, positive constants. Also, it is known that (Whittington *et al* 1983)

$$a_n(c) \leqslant (2dn)^c a_n(0) . \qquad (3.2)$$

If the bounds on $a_n(0)$ in theorem 2 are substituted into (3.1) and (3.2), then the following bounds are found on c-animals:

$$\mathrm{O}(n^{c-24\alpha})\mathrm{e}^{-\delta[\log(n-c)]^2}\lambda_0{}^n \leqslant a_n(c) \leqslant (2dn)^c \lambda_0{}^n . \qquad (3.3)$$

Of course, while we found interesting bounds on t_n in this paper, it is by no means a final word on this problem. The real challenge would be first to proof the analogous bounds for polygons (equation (1.2)) for trees, where the corrections to pure exponential growth are bound by powers of n. A proof that the critical exponent θ exists remains elusive in eight or less dimensions. So far, equation (1.5) is the strongest bound known.

Acknowledgment

S G Whittington was instrumental in the development of the ideas in this paper.

References

Adler J, Meir Y, Harris A B, Aharony A and Duarte J A M S 1988 *Phys. Rev.* D **38** 4941
Caracciolo S and Glaus U 1985 *J. Stat. Phys.* **41** 95
Glaus U 1985 *J. Phys. A: Math. Gen.* **18** L609
Gradshteyn I S and Ryznik I M 1965 *Table of Integrals, Series and Products* (London: Academic)
Hammersley J M 1961 *Proc. Camb. Phil. Soc.* **57** 516
—— 1962 *Proc. Camb. Phil. Soc.* **58** 235
Hammersley J M and Morton K W 1954 *J. R. Stat. Soc.* B **16** 23
Hammersley J M and Welsh D J A 1962 *Quart J. Math. Oxford* **13** 108
Hara T and Slade G 1991a *Bull. Amer. Math. Soc.* **25** 417
—— 1991b *J. Stat. Phys.* **59** 1469
—— 1992 *Preprint*
Hille E 1948 *Functional Analysis and Semi-Groups (Amer. Math. Soc. Colloq. Publ.)* vol 31 (New York: AMS)
Ishinabe T 1989 *J. Phys. A: Math. Gen.* **22** 4419
Kesten H 1964 *J. Math. Phys.* **5** 1128
Klein D J 1981 *J. Chem. Phys.* **75** 5186
Madras N 1991 *Random Walks, Brownian Motion and Interacting Particle Systems* ed R Durrett and H Kesten (Boston: Birkhauser)
Parisi G and Sourlas N 1981 *Phys. Rev. Lett.* **46** 871
Soteros C E and Whittington S G 1988 *J. Phys. A: Math. Gen.* **21** 2187
Whittington S G and Soteros C E 1991 Preprint, *Macromol. Rep.* submitted
Whittington S G, Torrie G M and Gaunt D S 1983 *J. Phys. A: Math. Gen.* **16** 1695